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Equatorial Ocean Dynamics in a Nonlinear Multimode Model

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Contents

List of Figures

1 Introduction

2 Derivation of the model equations

The full set of governing equations for an incompressible fluid in a rotating system under the Boussinesq and hydrostatic approximation is given by

$$\frac{\partial u}{\partial t} + \frac{\partial (u\,u)}{\partial x} + \frac{\partial (v\,u)}{\partial y} + \frac{\partial (w\,u)}{\partial z} - f\,v + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} - \frac{\partial}{\partial z}(A_v\,\frac{\partial u}{\partial z}) - A_h\,\vec{\nabla}_h^2\,u = 0 \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + \frac{\partial (u\,v)}{\partial x} + \frac{\partial (v\,v)}{\partial y} + \frac{\partial (w\,v)}{\partial z} + f\,u + \frac{1}{\rho_0}\frac{\partial p'}{\partial y} - \frac{\partial}{\partial z}(A_v\,\frac{\partial v}{\partial z}) - A_h\,\vec{\nabla}_h^2\,v = 0 \tag{2.1b}$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial (u\,\rho')}{\partial x} + \frac{\partial (v\,\rho')}{\partial y} + \frac{\partial (w\,\rho')}{\partial z} - \frac{\rho_0}{g}\,N^2\,w - \frac{\partial}{\partial z}(\kappa_v\,\frac{\partial \rho'}{\partial z}) - \kappa_h\,\vec{\nabla}_h^2\,\rho' = 0 \tag{2.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1d}$$

$$\frac{\partial p'}{\partial z} + g\,\rho' = 0. \tag{2.1e}$$

The coordinates (x, y, z, t) define a locally orthogonal system where t is the time and x measures the eastward, y the northward and z the upwards direction. The velocity is given accordingly, with u being the eastward, v the northward and w the upward component. For simplicity, the derivations in this chapter are presented in a Cartesian framework. Pressure p and density ρ are already separated into a background state that only depends on z and perturbations from that state

$$p(t, x, y, z) = \bar{p}(z) + p'(t, x, y, z) \tag{2.2}$$

$$\rho(t, x, y, z) = \bar{\rho}(z) + \rho'(t, x, y, z). \tag{2.3}$$

N is the *BruntVäisälä frequency*, measuring the frequency with which a fluid parcel would oscillate when it is displaced vertically as subject to the background density $\bar{\rho}$. It is given by

$$N^2(z) = -\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}, \tag{2.4}$$

where ρ_0 is a reference density and g the gravitational acceleration. The Coriolis force is accounted for with the latitude dependent Coriolis parameter f. The Cartesian framework thus implies the formulation of f on a $\beta-$plane. In the vicinity of the equator, $y = 0$, this means that

$$f = \beta\, y, \qquad \text{with} \qquad \beta = \frac{df}{dy}. \tag{2.5}$$

The horizontal and vertical eddy viscosity is denoted as A_h and A_v respectively, the thermal diffusivity as κ_h and κ_v respectively.

2.1 Modal decomposition

This section introduces the decomposition of the dynamical field described by equations (**??**) into *vertical normal modes* and *vertical structure functions.* In order to explain the underlying concept, nonlinear terms and viscosity will be neglected. For convenience in the following derivations, the perturbation pressure is substituted by the variable η defined as

$$\eta \equiv \frac{1}{g\,\rho_0}\, p'. \tag{2.6}$$

This variable is of units $[\eta] = m$ and corresponds to the surface elevation in a homogeneous shallow water analogue. The equations of motion then take the form

$$\frac{\partial u}{\partial t} - f\,v + g\,\frac{\partial \eta}{\partial x} = 0 \tag{2.7a}$$
$$\frac{\partial v}{\partial t} + f\,u + g\,\frac{\partial \eta}{\partial y} = 0 \tag{2.7b}$$
$$\frac{\partial \rho'}{\partial t} - \frac{\rho_0}{g}\,N^2\,w = 0 \tag{2.7c}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.7d}$$
$$\frac{\partial \eta}{\partial z} + \frac{1}{\rho_0}\,\rho' = 0. \tag{2.7e}$$

Inserting (**??**) in (**??**) eliminates ρ' and gives

$$w = -\frac{g}{N^2}\,\frac{\partial}{\partial z}\,\frac{\partial \eta}{\partial t}$$
$$\text{and} \qquad \frac{\partial w}{\partial z} = -g\,(\frac{\partial}{\partial z}\,\frac{1}{N^2}\,\frac{\partial}{\partial z})\,\frac{\partial \eta}{\partial t}. \tag{2.8}$$

With (**??**) and (**??**) w can be eliminated, ultimately resulting in three equations for three unknowns, u, v and η. By introducing the linear operator

$$\Lambda \equiv (\frac{\partial}{\partial z} \frac{1}{N^2} \frac{\partial}{\partial z}), \tag{2.9}$$

these can be written as

$$\frac{\partial u}{\partial t} - f\, v + g\, \frac{\partial \eta}{\partial x} = 0 \tag{2.10a}$$
$$\frac{\partial v}{\partial t} + f\, u + g\, \frac{\partial \eta}{\partial y} = 0 \tag{2.10b}$$
$$-\Lambda\, \frac{\partial \eta}{\partial t} + \frac{1}{g} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \tag{2.10c}$$

By not considering any vertical mixing or nonlinear terms, all vertical derivatives of the system are solely expressed by the linear operator Λ.

The eigenvalue problem for this operator

$$\Lambda\, \psi_n = \frac{d}{dz} \left(\frac{1}{N^2} \frac{d\psi_n}{dz} \right) = -\lambda_n\, \psi_n \qquad (n \in \mathbb{Z})\,, \tag{2.11}$$

together with the boundary conditions

$$\frac{1}{N^2} \frac{d\psi_n}{dz} = 0 \text{ at } z = 0, -H, \tag{2.12}$$

takes the form of a classical Sturm-Liouville problem. The boundary conditions are defined at the surface $z = 0$ and the bottom $z = -H$, where H is the uniform water depth. This assumes a flat bottom, neglecting any variable bottom. Note that the boundary conditions in (**??**) must also hold when multiplied by N^2 if the buoyancy frequency stays finite at the boundaries, as is the case for the profile used for this thesis.

As indicated by the indices n, this problem solves for an infinite, but countable set of orthogonal eigenfunctions. They only depend on z and can be normalized, such that the inner product defined below yields

$$\langle \psi_m | \psi_n \rangle \equiv \frac{1}{H} \int_{-H}^{0} \psi_m\, \psi_n dz = \delta_{m,n}, \tag{2.13}$$

where

$$\delta_{m,n} = \begin{cases} 1 & n = m \\ 0 & \text{otherwise.} \end{cases}$$

The countability allows for the eigenfunctions to be sorted by the value of their corresponding eigenvalue and certain modes can therefore be identified. The zeroth eigenfunction ψ_0 has the smallest eigenvalue with $\lambda_0 = 0$ and is by (**??**) depth independent. This mode is called the *barotropic mode* and requires special care. As will be seen in the following section $\lambda_0 = 0$ corresponds to an infinite gravity wave speed. Thus, the model developed for the purposes of this thesis only accounts for the *baroclinic* modes where $n > 0$.

The remaining solutions to the system described in (**??**) decouple in the vertical and can be expressed as linear combinations of the now ordered eigenfunctions of Λ.

$$u(x,y,z,t) = \sum_{n=1}^{\infty} \tilde{u}_n(x,y,t)\,\psi_n(z) \tag{2.14a}$$

$$v(x,y,z,t) = \sum_{n=1}^{\infty} \tilde{v}_n(x,y,t)\,\psi_n(z) \tag{2.14b}$$

$$\eta(x,y,z,t) = \sum_{n=1}^{\infty} \tilde{\eta}_n(x,y,t)\,\psi_n(z). \tag{2.14c}$$

The advantage is that all variables separate into the (z)-dependent eigenfunctions and the (x, y, t)-dependent coefficients. The coefficients $\tilde{u}_k$, $\tilde{v}_k$ and $\tilde{\eta}_k$ are defined through the projection of u,v and η onto the k^{th} eigenfunction by the operator

$$\langle\psi_k|\,[...] \equiv \frac{1}{H}\int_{-H}^{0} \psi_k\,[...]\,dz. \tag{2.15}$$

With the orthogonality condition given in (**??**), this means that

$$\tilde{u}_k(x,y,t) = \frac{1}{H}\int_{-H}^{0} u\,\psi_k\,dz \tag{2.16a}$$

$$\tilde{v}_k(x,y,t) = \frac{1}{H}\int_{-H}^{0} v\,\psi_k\,dz \tag{2.16b}$$

$$\tilde{\eta}_k(x,y,t) = \frac{1}{H}\int_{-H}^{0} \eta\,\psi_k\,dz. \tag{2.16c}$$

In the context presented here, the eigenfunctions are denoted as *vertical structure functions* or *vertical normal modes*.

2.2 Analogy to the Shallow Water Equations

When assuming an ocean without stratification ($N^2 = 0$), the governing equations can be averaged over depth to form the *Shallow Water Equations* (SWEs). They describe the motion of a layer of incompressible fluid of homogeneous density (?, ?), subject to gravity and surface pressure. The SWEs are widely used in the field of atmosphere and ocean dynamics and form a useful tool for studying wave propagation. However, the homogeneous density field does not allow for any variations with depth. This constraint is relaxed in the following. Sets of SWEs for all vertical normal modes of a depth dependent stratification are derived by using the concepts described in the previous section.

Solving the Sturm-Liouville problem (**??**) gives the vertical structure functions and their respective eigenvalues with units $[\lambda_n] = s^2\, m^{-2}$. It is useful to introduce the mode dependent *gravity wave speed* as

$$c_n = \sqrt{\frac{1}{\lambda_n}}. \tag{2.17}$$

This choice is motivated by its analogy to the gravity wave speed c in the homogeneous SWEs. The analogy further encourages the definition of the *equivalent depth* through

$$H_n = \frac{1}{g\,\lambda_n}, \tag{2.18}$$

corresponding to the depth of the homogeneous ocean layer in the SWEs. It is worth pointing out the implication for the barotropic mode again. Having $\lambda = 0$ leads to an infinitely deep equivalent depth and an infinitely fast gravity wave speed. The model formulation given here neglects the barotropic mode.

The equations (**??**) are now projected onto the k^{th} mode by applying the operator defined in (**??**). The equations (**??**) then take the form

$$\frac{1}{H}\int_{-H}^{0}\left[\frac{\partial u}{\partial t} - f\,v + g\,\frac{\partial \eta}{\partial x}\right]\,\psi_k\,dz = 0 \tag{2.19a}$$

$$\frac{1}{H}\int_{-H}^{0}\left[\frac{\partial v}{\partial t} + f\,u + g\,\frac{\partial \eta}{\partial y}\right]\,\psi_k\,dz = 0 \tag{2.19b}$$

$$\frac{1}{H}\int_{-H}^{0}\left[-\Lambda\,\frac{\partial \eta}{\partial t} + \frac{1}{g}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]\,\psi_k\,dz = 0. \tag{2.19c}$$

With a double integration-by-parts subject to proper boundary conditions (discussed in section **??**) the operator Λ can be shifted to ψ_k and replaced by the eigenvalue according to equation (**??**). Next, all constants and x-, y- or t-derivatives can be pulled out of the integral. Only the k^{th} mode of each expanded variable is non-zero, selecting only the k^{th} expansion coefficients through (**??**). The projection of (**??**) decouples in the vertical and takes the form

$$\frac{\partial \tilde{u}_k}{\partial t} - f\,\tilde{v}_k + g\,\frac{\partial \tilde{\eta}_k}{\partial x} = 0 \tag{2.20a}$$

$$\frac{\partial \tilde{v}_k}{\partial t} + f\,\tilde{u}_k + g\,\frac{\partial \tilde{\eta}_k}{\partial y} = 0 \tag{2.20b}$$

$$\frac{\partial \tilde{\eta}_k}{\partial t} + H_k\left(\frac{\partial \tilde{u}_k}{\partial x} + \frac{\partial \tilde{v}_k}{\partial y}\right) = 0. \tag{2.20c}$$

Equations (**??**) correspond to the SWEs for the k^{th} mode with the water depth H being replaced by the equivalent depth H_k. The choice of k is arbitrary, so equations (**??**) hold for every vertical normal mode independently.

2.3 Diagnostic variables

The two diagnostic variables ρ' and w cannot be expanded straightforwardly, since their dependency on u, v and η through (**??**) and (**??**) involves vertical derivatives of infinite sums. This is a delicate matter which requires care, because derivatives of infinite sequences do not generally converge (see theorem 7.17 in ? (?)). In order to show that the expansions of the diagnostic variables do converge, a similarly diagnostic eigenvalue problem is introduced. By taking the vertical derivative of (**??**), a second Sturm-Liouville problem can be formulated as

$$\frac{d^2}{dz^2}\left(-\frac{g}{N^2}\,\frac{d\psi_n}{dz}\right) = -\lambda_n\,N^2\left(-\frac{g}{N^2}\,\frac{d\psi_n}{dz}\right) \quad (n \in \mathbb{Z})\,. \tag{2.21}$$

The new eigenfunctions

$$\chi_n = -\frac{g}{N^2}\frac{d\psi_n}{dz}, \tag{2.22}$$

are scaled with the, at this point, arbitrary constant factor $-g$, which is taken to be the acceleration due to gravity. This makes the eigenfunctions unitless. The boundary conditions follow naturally as

$$\chi_n = 0 \text{ at } z = 0, -H. \tag{2.23}$$

It further follows from (**??**) and (**??**) that

$$\frac{d\chi_n}{dz} = \frac{1}{H_n}\psi_n. \tag{2.24}$$

Following Sturm-Liouville theory, the eigenfunctions again form an orthogonal base and can be normalized consistently with (**??**), such that

$$\langle \chi_m | \chi_n \rangle \equiv \frac{H_n}{H\,g} \int_{-H}^{0} N^2\, \chi_m\, \chi_n dz = \delta_{m,n}. \tag{2.25}$$

Similarly to (**??**), this defines an operator which projects onto the k^{th} χ-mode as

$$\langle \psi_k | \, [...] \equiv \frac{H_k}{H\,g} \int_{-H}^{0} N^2\, \chi_k \, [...] \, dz. \tag{2.26}$$

It is further convenient to introduce the *isopycnal displacement* as

$$h = \frac{g}{\rho_0} \frac{\rho'}{N^2}, \tag{2.27}$$

which measures the vertical displacement of isopycnals from the background state caused by perturbations in the density field. With help of equations (**??**) and(**??**) it is shown in Appendix **??** that w and h can be written as linear combinations of the χ-modes. Expansions with the right coefficients are then given by

$$h(x,y,z,t) = \sum_{n=1}^{\infty} \tilde{\eta}_n\, \chi_n. \tag{2.28a}$$

$$w(x,y,z,t) = \sum_{n=1}^{\infty} \tilde{w}_n \chi_n, \tag{2.28b}$$

with

$$\tilde{w}_n = -H_n \left(\frac{\partial \tilde{u}_n}{\partial x} + \frac{\partial \tilde{v}_n}{\partial y} \right). \tag{2.29}$$

Equations (**??**) and (**??**) also allow for an expansion of the perturbation density as

$$\rho'(x,y,z,t) = -\frac{\rho_0\, N^2}{g} \sum_{n=1}^{\infty} \tilde{\eta}_n(x,y,t)\, \chi_n. \tag{2.30}$$

While the modal velocities $\tilde{u}_k$ and $\tilde{v}_k$ can be interpreted plainly as zonal and meridional velocity fields associated with the vertical structure defined by ψ_k, $\tilde{\eta}_k$ can be interpreted simultaneously as the k^{th} expansion coefficient of the isopycnal displacement h and the scaled pressure perturbation η, depending on the used set of eigenfunctions.

The explicit relation between ψ_k and χ_k in (**??**) mirrors the dependency between the diagnostic variables w, h, ρ' and the prognostic variables of the system (**??**). It should be noted that both eigenvalue problems can be derived from (**??**) and equivalently yield the structure functions. Both formulations are used in the literature (see e.g. ? (?) and ? (?)).

Another pair of diagnostic variables is the kinetic energy density E_{kin} and the available potential energy density E_{pot}. When averaged vertically, they take the form

$$\bar{E}_{kin} = \frac{1}{H} \int_{-H}^{0} \frac{\rho_0}{2} \left(u^2 + v^2\right) \, dz \tag{2.31a}$$

$$\bar{E}_{pot} = \frac{1}{H} \int_{-H}^{0} \frac{\rho_0}{2} N^2 h^2 \, dz. \tag{2.31b}$$

Note, that these expressions need to be integrated over a volume in order to yield the kinetic energy and potential energy contained in that volume, and so include an additional area integral. Inserting the expansions (**??**), (**??**) and (**??**) and identifying the orthogonality conditions from equations (**??**), (**??**) yields $\bar{E}_{kin}$ and $\bar{E}_{pot}$ as sums over the squared variables in mode space

$$\bar{E}_{kin} = \frac{\rho_0}{2} \sum_{n=1}^{\infty} \left(\tilde{u}_n^2 + \tilde{v}_n^2\right) \tag{2.32a}$$

$$\bar{E}_{pot} = \frac{\rho_0}{2} \sum_{n=1}^{\infty} \frac{g}{H_n} \tilde{h}_n^2. \tag{2.32b}$$

The result therefore allows for the measure of the vertically-averaged energy density captured by each individual mode. For more detailed information see ? (?).

2.4 Boundary conditions

The vertical boundary condition imposed on χ_n (**??**) directly determines w and ρ' at the bottom and surface through (**??**) and (**??**). It follows at $z = 0$ (**surface**) and $z = -H$ (**bottom**) that

$$w = 0. \tag{2.33a}$$
$$\rho' = 0 \tag{2.33b}$$

These present hidden assumptions made to formulate the boundary condition of the Sturm-Liouville problem.

i) The conditions on w (**??**) is the rigid lid approximation. Thinking of what we observe in the real ocean this seems rather unrealistic. However, the only mode strongly affected by this is the barotropic mode ($n = 0$) (?, ?). The barotropic mode is not considered in the model presented here and for all baroclinic modes this assumption is very good.

ii) The boundary condition for ρ' (**??**) prohibits any temporal changes from the mean state density at the surface or bottom over the whole basin and thus does not allow for the generation of a responsive sea surface temperature field (?, ?). This issue is a serious limitation and can be problematic when vertical velocities extend close to the surface, namely regions of up- and downwelling. Previous authors found ways of relaxing this constraint (?, ?), but within the scope of this thesis the boundary conditions on ρ' remain as stated.

Boundary conditions for the horizontal velocities are defined through their vertical derivative. At the **surface**, the vertical shear of the horizontal velocities is given by the downward mixing of the wind stress, namely

$$A_v \frac{\partial u}{\partial z} = \frac{\tau_x}{\rho_0} \tag{2.34a}$$
$$A_v \frac{\partial v}{\partial z} = \frac{\tau_y}{\rho_0}. \tag{2.34b}$$

No stress is assumed at the **bottom**, resulting in

$$\frac{\partial u}{\partial z} = 0 \tag{2.35a}$$
$$\frac{\partial v}{\partial z} = 0. \tag{2.35b}$$

The conditions posed in (**??**) and (**??**) provide an example for the issue of derivatives of infinite sums mentioned in the previous section. If one would insert the expansions (**??**), (**??**) and simply apply the vertical derivatives to all terms of the infinite sums, the boundary conditions on ψ_n would prohibit any vertical shear at the surface. This would also prohibit any wind forcing, a limitation which cannot be accepted for a model of the equatorial ocean. Fortunately, the infinite number of eigenfunctions allows for both, the expansion of u and v and their surface boundary conditions to be satisfied.

2.5 Mode-to-Mode interaction

The equations (**??**) state that the flow field of every vertical normal mode is independently following the SWEs with a mode dependent equivalent depth H_k replacing the water depth. This is also the only mode dependent parameter entering the system, which determines the speed with which gravity waves propagate by (**??**). This property was exploited half a century ago to study the oceanic response to the onset of winds for individual vertical normal modes (?, ?).

However, the uncoupled system does not allow for any mode-to-mode interaction and thereby prohibits the transfer of energy between structures of different vertical scale. In this section this constraint is relaxed by reintroducing the viscosity and nonlinear terms. These terms contain vertical derivative operators that usually differ from Λ and do therefore not decouple in the projection onto vertical normal modes.

Equations (**??**) and (**??**) can now be projected onto ψ_k, equation (**??**) onto χ_k. The latter is first multiplied by $\frac{\rho_0}{g\,N^2}$ to form an equation for h

$$\begin{aligned}\frac{\partial h}{\partial t} - w - \kappa_h\,\vec{\nabla}_h^2\,h = \frac{1}{N^2}\frac{\partial}{\partial z}(\kappa_v\,\frac{\partial N^2\,h}{\partial z})\ldots \\ \ldots - \frac{1}{N^2}\left(\frac{\partial(u\,h\,N^2)}{\partial x} + \frac{\partial(v\,h\,N^2)}{\partial y} + \frac{\partial(w\,h\,N^2)}{\partial z}\right).\end{aligned} \tag{2.36}$$

The projections of the governing equations onto the k^{th} mode with the operators (**??**) and (**??**) then take the form

$$\frac{1}{H}\int_{-H}^{0}\left[\frac{\partial u}{\partial t} - f\,v + g\,\frac{\partial \eta}{\partial x} - A_h\,\vec{\nabla}_h^2\,u\right]\psi_k = \frac{1}{H}\int_{-H}^{0}\left(\nu^{(u)} - \zeta^{(u)}\right)\,\psi_k\,dz \quad (2.37a)$$

$$\frac{1}{H}\int_{-H}^{0}\left[\frac{\partial v}{\partial t} + f\,u + g\,\frac{\partial \eta}{\partial y} - A_h\,\vec{\nabla}_h^2\,v\right]\psi_k = \frac{1}{H}\int_{-H}^{0}\left(\nu^{(v)} - \zeta^{(v)}\right)\,\psi_k\,dz \quad (2.37b)$$

$$\frac{H_k}{H\,g}\int_{-H}^{0}\left[\frac{\partial h}{\partial t} - w - \kappa_h\,\vec{\nabla}_h^2\,h\right]N^2\,\chi_k\,dz. = \frac{H_k}{H\,g}\int_{-H}^{0}\left(\nu^{(N^2\,h)} - \zeta^{(N^2\,h)}\right)\,\chi_k\,dz. \quad (2.37c)$$

where the vertical mixing and nonlinear advection of a variable are denoted as

$$\nu^{(\dots)} = \frac{\partial}{\partial z}\left(A_v/\kappa_v\,\frac{\partial \dots}{\partial z}\right) \quad (2.38)$$

and

$$\zeta^{(\dots)} = \frac{\partial(u\,\dots)}{\partial x} + \frac{\partial(v\,\dots)}{\partial y} + \frac{\partial(w\,\dots)}{\partial z}. \quad (2.39)$$

As in Section **??**, the orthogonality of (**??**) and (**??**) allows to project the left-hand-side of (**??**) entirely on the k^{th} mode, resulting in

$$\frac{\partial \tilde{u}_k}{\partial t} - f\,\tilde{v}_k + g\,\frac{\partial \tilde{\eta}_k}{\partial x} - A_h\,\vec{\nabla}_h^2\,\tilde{u}_k = \frac{1}{H}\int_{-H}^{0}\left(\nu^{(u)} - \zeta^{(u)}\right)\,\psi_k\,dz \quad (2.40a)$$

$$\frac{\partial \tilde{v}_k}{\partial t} + f\,\tilde{u}_k + g\,\frac{\partial \tilde{\eta}_k}{\partial y} - A_h\,\vec{\nabla}_h^2\,\tilde{v}_k = \frac{1}{H}\int_{-H}^{0}\left(\nu^{(v)} - \zeta^{(v)}\right)\,\psi_k\,dz \quad (2.40b)$$

$$\frac{\partial \tilde{\eta}_k}{\partial t} - \tilde{w}_k - \kappa_h\,\vec{\nabla}_h^2\tilde{\eta}_k = \frac{H_k}{H\,g}\int_{-H}^{0}\left(\nu^{(N^2\,h)} - \zeta^{(N^2\,h)}\right)\,\chi_k\,dz. \quad (2.40c)$$

The right-hand-side still contains vertical derivatives and does therefore not necessarily decouple. The shape of the coupling through vertical mixing and nonlinear advection in mode-space is demonstrated in the following subsections.

2.5.1 Vertical mixing

Before expanding the vertical mixing terms on the right-hand-side of (**??**), the result can be simplified with integration-by-parts using the boundary conditions presented in section**??**.

$$\frac{1}{H}\int_{-H}^{0}\frac{\partial}{\partial z}\left(A_v\,\frac{\partial u}{\partial z}\right)\psi_k\,dz = \frac{\tau_x}{\rho_0\,H}\,\psi_k(0) - \frac{1}{H}\int_{-H}^{0} A_v\,\frac{\partial u}{\partial z}\,\frac{\partial \psi_k}{\partial z}\,dz \tag{2.41a}$$

$$\frac{1}{H}\int_{-H}^{0}\frac{\partial}{\partial z}\left(A_v\,\frac{\partial v}{\partial z}\right)\psi_k\,dz = \frac{\tau_y}{\rho_0\,H}\,\psi_k(0) - \frac{1}{H}\int_{-H}^{0} A_v\,\frac{\partial v}{\partial z}\,\frac{\partial \psi_k}{\partial z}\,dz \tag{2.41b}$$

$$\begin{aligned}\frac{H_k}{H\,g}\int_{-H}^{0}\frac{\partial}{\partial z}\left(\kappa_v\,\frac{\partial(N^2\,h)}{\partial z}\right)\chi_k\,dz &= -\frac{1}{H\,g}\int_{-H}^{0}\kappa_v\,\frac{\partial(N^2\,h)}{\partial z}\psi_k\,dz.\\ &= \frac{1}{H\,g}\int_{-H}^{0} N^2\,h\,\frac{\partial\left(\kappa_v\,\psi_k\right)}{\partial z}\,dz.\end{aligned} \tag{2.41c}$$

Note that for the vertical mixing of perturbation density (**??**) was inserted and the integration-by-parts was performed twice. By introducing the vertical mixing tensors

$$\mathcal{P}_{n,k} = \frac{1}{H}\int_{-H}^{0} A_v\,\frac{\partial \psi_n}{\partial z}\,\frac{\partial \psi_k}{\partial z}\,dz, \tag{2.42a}$$

$$\mathcal{Q}_{n,k} = \frac{1}{H}\int_{-H}^{0} \frac{\partial \psi_n}{\partial z}\,\frac{\partial\left(\kappa_v\,\psi_k\right)}{\partial z}\,dz, \tag{2.42b}$$

the expansions in the right-hand-side of (**??**) take the form

$$\ldots = \frac{\tau_x}{\rho_0\,H}\,\psi_k(0) - \sum_{n=1}^{\infty}\tilde{u}_n\,\mathcal{P}_{n,k} \tag{2.43a}$$

$$\ldots = \frac{\tau_y}{\rho_0\,H}\,\psi_k(0) - \sum_{n=1}^{\infty}\tilde{v}_n\,\mathcal{P}_{n,k} \tag{2.43b}$$

$$\ldots = -\sum_{n=1}^{\infty}\tilde{\eta}_n\,\mathcal{Q}_{n,k}. \tag{2.43c}$$

The wind forcing appears as a natural consequence of the surface boundary conditions for the vertical mixing of momentum. The vertical mixing allows for mode-to-mode interactions through the tensors $\mathcal{P}$ and $\mathcal{Q}$. When assuming time independent viscosity coefficients A_v and κ_v, these tensors become constant in time and thus enter the model equations simply as a parameter.

Two choices for the vertical mixing coefficients are used for the experiments presented in Section **??**. The resulting tensors $\mathcal{P}$ and $\mathcal{Q}$ can be further simplified, as shown in the following:

I) Uniform vertical mixing

When assuming the viscosity coefficients A_v and κ_v to be depth-independent, they can be pulled out of derivatives and integrals. The mixing tensors $\mathcal{P}$ and $\mathcal{Q}$ then have the same shape. They simply scale through their coefficients such that

$$\mathcal{P}_{n,k} = \frac{A_v}{H} \int_{-H}^{0} \frac{\partial \psi_n}{\partial z} \frac{\partial \psi_k}{\partial z} \, dz, = \frac{A_v}{\kappa_v} \mathcal{Q}_{n,k}. \tag{2.44}$$

II) McCrearys vertical mixing

As mentioned earlier, the vertical mixing terms do not decouple in the modal decomposition, because its applied derivative operator is not of form Λ in **??**. For this reason ? (?) chose special, depth-dependent vertical mixing coefficients as

$$A_v = \frac{A}{N^2} \tag{2.45a}$$

$$\kappa_v = \frac{B}{N^2}, \tag{2.45b}$$

where A and B are constants determining the magnitude of the mixing. Additionally, he changed the term for vertical mixing of density in a way that

$$\frac{\partial}{\partial z} \left(\kappa_v \frac{\partial \rho'}{\partial z} \right) \rightarrow \frac{\partial^2}{\partial z^2} \left(\kappa_v \, \rho' \right). \tag{2.46}$$

The vertical mixing tensors in (**??**) and (**??**) then take the form

$$\mathcal{P}_{n,k} = \frac{1}{H} \int_{-H}^{0} \frac{A}{N^2} \frac{\partial \psi_n}{\partial z} \frac{\partial \psi_k}{\partial z} \, dz \tag{2.47a}$$

$$\mathcal{Q}_{n,k} = \frac{1}{H} \int_{-H}^{0} \frac{B}{N^2} \frac{\partial \psi_n}{\partial z} \frac{\partial \psi_k}{\partial z} \, dz. \tag{2.47b}$$

Another integration-by-parts and inserting the eigenvalue equation (**??**) yields

$$\mathcal{P}_{n,k} = \frac{A}{H_k \, g} \frac{1}{H} \int_{-H}^{0} \psi_n \, \psi_k \, dz = \frac{A}{H_k \, g} \delta_{n,k} \tag{2.48a}$$

$$\mathcal{Q}_{n,k} = \frac{B}{H_k \, g} \frac{1}{H} \int_{-H}^{0} \psi_n \, \psi_k \, dz = \frac{B}{H_k \, g} \delta_{n,k}, \tag{2.48b}$$

where the orthogonality stated by (**??**) was used. The tensors are only left with entries on their main diagonal and the sums in the right-hand-side of (**??**) reduce to

$$\ldots = \frac{\tau_x}{\rho_0 H}\,\psi_k(0) - \frac{A}{H_k\,g}\,\tilde{u}_k \tag{2.49a}$$

$$\ldots = \frac{\tau_y}{\rho_0 H}\,\psi_k(0) - \frac{A}{H_k\,g}\,\tilde{v}_k \tag{2.49b}$$

$$\ldots = -\frac{B}{H_k\,g}\,\tilde{\eta}_k. \tag{2.49c}$$

The linear model equations fully decouple and form independent sets of SWEs, with the equivalent depth H_k, uniform wind forcing $\frac{\tau}{\rho_0 H}\,\psi_k(0)$ and linear damping terms with coefficients $\frac{A}{H_k\,g}$, $\frac{B}{H_k\,g}$. The choice of the coefficients in (**??**) is named after J.P. McCreary, since he used this formulation to study the equatorial current system with a linear multimode model. It should be noted however, that preceding authors already introduced these forms to study internal waves (?, ?). Besides the mathematical convenience this formulation has in the projected equations, it is also intuitive to have more intense vertical mixing where the stratification is weak. There is also observational support for an inverse proportionality of the vertical mixing coefficient to the buoyancy frequency (?, ?). However, this implies intense vertical mixing in the deep ocean, which is shown to be weak for the equatorial ocean (?, ?).

2.5.2 Advection

The nonlinear advection terms permit mode-to-mode interactions for two reasons. They contain vertical derivative operators differing from Λ in the vertical advection and products of variables subject to the expansions. Nevertheless, a similar simplification through integration-by-parts, as for the vertical mixing, can be applied for the vertical advection terms. Here, the additional term from the integration-by-parts vanishes due to the boundary conditions on $w = 0$ at $z = 0$ and $z = -H$. The projected advection then takes the shape

$$\frac{1}{H}\int_{-H}^{0}\zeta^{(u)}\,\psi_k\,dz = \frac{1}{H}\int_{-H}^{0}\left(\frac{\partial(u\,u)}{\partial x}+\frac{\partial(v\,u)}{\partial y}\right)\psi_k-(w\,u)\,\frac{\partial\psi_k}{\partial z}\,dz \tag{2.50a}$$

$$\frac{1}{H}\int_{-H}^{0}\zeta^{(v)}\,\psi_k\,dz = \frac{1}{H}\int_{-H}^{0}\left(\frac{\partial(u\,v)}{\partial x}+\frac{\partial(v\,v)}{\partial y}\right)\psi_k-(w\,v)\,\frac{\partial\psi_k}{\partial z}\,dz \tag{2.50b}$$

$$\frac{H_k}{H\,g}\int_{-H}^{0}\zeta^{(N^2\,h)}\,\chi_k dz = \frac{H_k}{H\,g}\int_{-H}^{0}\left(\frac{\partial(u\,h)}{\partial x}+\frac{\partial(v\,h)}{\partial y}\right)N^2\,\chi_k\dots$$

$$\dots-\frac{1}{H_k}(w\,h)\,N^2\,\psi_k\,dz. \tag{2.50c}$$

The relation between the two eigenfunctions in equation (??) was used in the vertical advection of density term. Defining the advection tensors

$$\mathcal{R}_{n,m,k} = \frac{1}{H}\int_{-H}^{0}\psi_n\,\psi_m\,\psi_k\,dz \tag{2.51a}$$

$$\begin{aligned}\mathcal{S}_{n,m,k} &= \frac{g}{H}\int_{-H}^{0}\frac{1}{N^2}\,\frac{\partial\psi_n}{\partial z}\,\psi_m\,\frac{\partial\psi_k}{\partial z}\,dz,\\ &= \frac{1}{H\,g}\int_{-H}^{0}N^2\,\chi_n\,\psi_m\,\chi_k\,dz,\end{aligned} \tag{2.51b}$$

and inserting the expansions for all variables in the right-hand-side of (??) leads to the projected terms for advection of zonal momentum $\mathcal{U}$, meridional momentum $\mathcal{V}$ and density perturbation $\mathcal{D}$ as

$$\mathcal{U}_k \equiv \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\partial(\tilde{u}_n\,\tilde{u}_m)}{\partial x}+\frac{\partial(\tilde{v}_n\,\tilde{u}_m)}{\partial y}\right)\mathcal{R}_{n,m,k}+(\tilde{w}_n\,\tilde{u}_m)\,\mathcal{S}_{n,m,k} \tag{2.52a}$$

$$\mathcal{V}_k \equiv \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\partial(\tilde{u}_n\,\tilde{v}_m)}{\partial x}+\frac{\partial(\tilde{v}_n\,\tilde{v}_m)}{\partial y}\right)\mathcal{R}_{n,m,k}+(\tilde{w}_n\,\tilde{v}_m)\,\mathcal{S}_{n,m,k} \tag{2.52b}$$

$$\mathcal{D}_k \equiv \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}H_k\left(\frac{\partial(\tilde{u}_n\,\tilde{\eta}_m)}{\partial x}+\frac{\partial(\tilde{v}_n\,\tilde{\eta}_m)}{\partial y}\right)\mathcal{S}_{m,n,k}-(\tilde{w}_n\,\tilde{\eta}_m)\,\mathcal{S}_{m,k,n}. \tag{2.52c}$$

2.6 The model equations

All previous derivations in this chapter finally allow for the formulation of the governing equation for an incompressible fluid in a rotating system under the Boussinesq and hydrostatic approximation (??) in mode space. When the boundary conditions from section **??** and a flat bottom are assumed, both formulations in physical space and mode space are equivalent. The transformation from physical to mode space is given by the expansions in (**??**), (**??**) and (**??**), which replaces the vertical coordinate axis with the infinite set of vertical normal modes. The dynamics for the k^{th} mode are then given by

$$\frac{\partial \tilde{u}_k}{\partial t} - f\,\tilde{v}_k + g\,\frac{\partial \tilde{\eta}_k}{\partial x} - A_h\,\vec{\nabla}_h^2\,\tilde{u}_k = \frac{\tau_x}{\rho_0\,H}\,\psi_k(0) - \sum_{n=1}^{\infty} \tilde{u}_n\,\mathcal{P}_{n,k} - \mathcal{U}_k \tag{2.53a}$$

$$\frac{\partial \tilde{v}_k}{\partial t} + f\,\tilde{u}_k + g\,\frac{\partial \tilde{\eta}_k}{\partial y} - A_h\,\vec{\nabla}_h^2\,\tilde{v}_k = \frac{\tau_y}{\rho_0\,H}\,\psi_k(0) - \sum_{n=1}^{\infty} \tilde{v}_n\,\mathcal{P}_{n,k} - \mathcal{V}_k \tag{2.53b}$$

$$\frac{\partial \tilde{\eta}_k}{\partial t} + H_k\left(\frac{\partial \tilde{u}_k}{\partial x} + \frac{\partial \tilde{v}_k}{\partial y}\right) - \kappa_h\,\vec{\nabla}_h^2\tilde{\eta}_k = -\sum_{n=1}^{\infty} \tilde{\eta}_n\,\mathcal{Q}_{n,k} - \mathcal{D}_k. \tag{2.53c}$$

In this work, the equations (**??**) are solved numerically for different wind forcings τ and vertical mixing tensors $\mathcal{P}$. The necessary numerical tools are described in the following chapter.

3 Nonlinear Multimode Model

The set of equations (**??**) forms the nonlinear multimode model derived and implemented in the course of this thesis. Although it describes a particular vertical normal mode with the discrete index k, the infinite set of eigenfunctions used in the expansion of variables (**??**) marks the problem as continuous. This is also mirrored in the equivalence to the clearly continuous equations in (**??**). The defined problem cannot be solved analytically, which motivates the implementation of the numerical nonlinear multimode model described in this chapter. To solve any physical problem numerically, the domain defined by all coordinate axes needs to be discretized. This is carried out in the following sections.

3.1 Discretization of the model domain

3.1.1 Horizontal

The model state is given by the three prognostic variables $\tilde{u}_k$, $\tilde{v}_k$ and $\tilde{\eta}_k$, which only depend on x, y and t. ? (?) studied possible horizontal grid arrangements of these variables for the problem defined in (**??**) and their implications for the representation of inertia-gravity waves. The *Arakawa C-grid* presented in Figure **??** is favorable under the condition that

$$\frac{\sqrt{H_k\, g}}{f} > \frac{\max\left(\Delta x, \Delta y\right)}{2}, \tag{3.1}$$

because it retains the positive group velocities for inertia-gravity waves with positive wave number. Therefore, the C-grid is also commonly used in various general circulation models (?, ?).

The grids are referenced to the q-grid, which is defined on the zonal and meridional indices

$$i = 0, \ldots, I-1 \qquad \text{and} \qquad j = 0, \ldots, J-1, \tag{3.2}$$

where I and J are the total number of grid points in the zonal and meridional direction. The $q-$grid does not hold the state of any variable, but is nevertheless useful for notation later on. The $u-$, $v-$ or $\eta-$grid is then obtained by shifting the $q-$grid by half its grid spacing in zonal, meridional or both directions. To account for this in notation, half indices are introduced such that the state defined in grid box (i, j) is written as

$$\left[u, v, \eta\right]_{i,j} = \left[u_{i,j+1/2}, v_{i+1/2,j}, \eta_{i+1/2,j+1/2}\right]. \tag{3.3}$$

One could argue that this configuration is not ideal for baroclinic models as

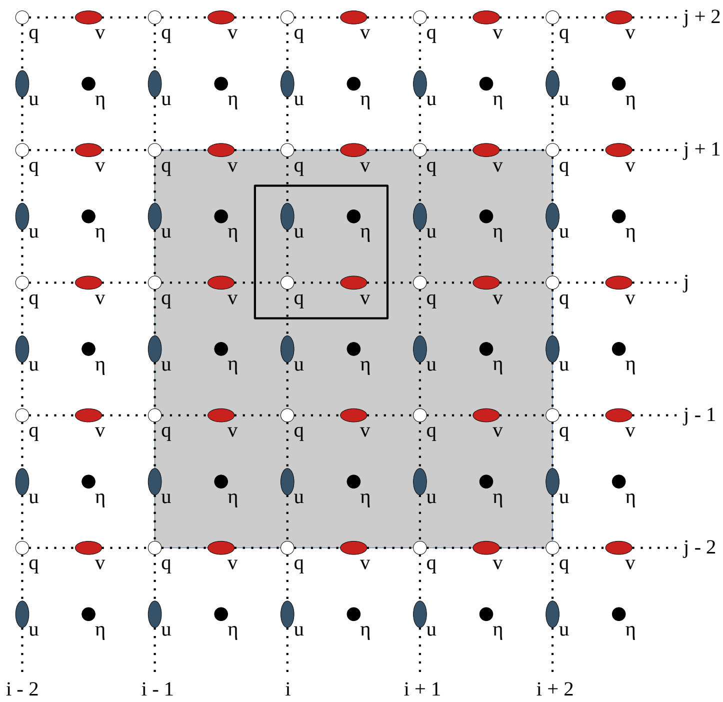

Figure 1: The Arakawa C-grid configuration. An exemplary ocean basin is shown as the shaded area. Lateral boundaries are defined on the points of normal velocity to prohibit flow in or out of them. The square box indicates all points of grid box (i, j), namely q, u, v and η.

presented here, since higher baroclinic modes demand a smaller horizontal grid spacing through (**??**). However, the phenomena studied with the model in this work are equatorial dynamics. The *Rossby radius of deformation*, given by the left-hand-side of (**??**), has its maximum in the vicinity of the equator and the condition on the horizontal grid spacing is easily fulfilled. The derivatives in (**??**) are approximated by centered finite differencing. For a staggered grid with indexing as shown in Figure **??**, the centered differencing operator applied to a variable α takes the form

$$\delta_i[\alpha] = \alpha_{i+1/2} - \alpha_{i-1/2} \tag{3.4a}$$

$$\delta_j[\alpha] = \alpha_{j+1/2} - \alpha_{j-1/2}. \tag{3.4b}$$

This is convenient for the terms found in (**??**), since all derivatives can be evaluated straightforwardly using centered differencing. For example, the zonal pressure gradient appears in the differential equation for u and accordingly its discrete difference in the C-grid evaluates on u-grid points. This is not the case for the differences in the advection terms, where some variables need to be averaged onto neighboring grid points with the average operator defined as

$$\overline{\alpha}^i = \frac{\alpha_{i+1/2} + \alpha_{i-1/2}}{2} \tag{3.5a}$$

$$\overline{\alpha}^j = \frac{\alpha_{j+1/2} + \alpha_{j-1/2}}{2}. \tag{3.5b}$$

Although the schematic in Figure **??** depicts a rectangular basin with a uniform grid spacing everywhere, this could be a poor choice to simulate the flow of fluids on a sphere. In order to allow for spherical coordinates, the model is formulated in *curvilinear coordinates*. The coordinate axes can thus be curved, as long as they are locally orthogonal to each other. All operators containing derivatives must then take the deformation of the grid with respect to Cartesian coordinates into account. This is defined through the horizontal grid spacing

$$dx_{i,j} = \delta_i[x_j] \tag{3.6a}$$

$$dy_{i,j} = \delta_j[y_i]. \tag{3.6b}$$

For simplicity in the notation, the half-indices of the horizontal grid spacing are replaced by the variable around which the difference is defined. All terms in (**??**) can then be horizontally discretized with the following operators acting on the scalar variable η or the vector field $\vec{u}$

$$\vec{\nabla}_h\, \eta = \frac{\delta_i[\eta]}{dx_u}\vec{e}_x + \frac{\delta_j[\eta]}{dy_v}\vec{e}_y \tag{3.7a}$$

$$\vec{\nabla}^2\, \eta = \frac{1}{dx_\eta\, dy_\eta} \left(\delta_{i+1/2}\left[\frac{dy_u}{dx_u}\,\delta_i[\eta]\right] + \delta_{j+1/2}\left[\frac{dx_v}{dy_v}\,\delta_j[\eta]\right]\right) \tag{3.7b}$$

$$\vec{\nabla}_h \cdot \vec{u} = \frac{1}{dx_\eta\, dy_\eta} \left(\delta_{i+1/2}\left[dy_u\, u\right] + \delta_{j+1/2}\left[dx_v\, v\right]\right) \tag{3.7c}$$

$$\begin{aligned}\vec{\nabla}^2\, \vec{u} = &\frac{1}{dx_u\, dy_u} \left(\delta_i\left[\frac{dy_\eta}{dx_\eta}\,\delta_i[u]\right] + \delta_{j+1/2}\left[\frac{dx_q}{dy_q}\,\delta_j[u]\right]\right)\vec{e}_x + \dots \\ &\dots + \frac{1}{dx_v\, dy_v} \left(\delta_{i+1/2}\left[\frac{dy_q}{dx_q}\,\delta_i[v]\right] + \delta_j\left[\frac{dx_\eta}{dy_\eta}\,\delta_j[v]\right]\right)\vec{e}_y.\end{aligned} \tag{3.7d}$$

This formulation follows the horizontal discretization of the NEMO model (?, ?). A detailed description of all discretized terms in (**??**) can be found in the appendix.

The horizontal domain for the presented experiments is defined on a spherical, regular latitude-longitude-grid with a 1/4° resolution. As stated earlier, the model equations (**??**) are derived in Cartesian coordinates and the transformation

$$(x, y) \rightarrow (\lambda, \phi) \tag{3.8}$$

is taken care of through the deformation of the horizontal grid spacing $(d\lambda, d\phi) = (1/4°, 1/4°)$ to (dx, dy), where λ is the longitude and ϕ the latitude. For a regular ϕ-λ-grid, this is given by

$$dy_{i,j} = \frac{d\phi_{i,j}}{180°}\,\pi\, r \approx 27.8\, km \tag{3.9a}$$

$$dx_{i,j} = \frac{d\lambda_{i,j}}{180°}\,\pi\, r \cos\phi_{i,j} \approx 27.8\, km \times \cos\phi_{i,j}, \tag{3.9b}$$

where $r \approx 6371\, km$ is the approximate radius of the Earth. The domain is meridionally centered around the v-, q-points defined on the equator, extending from $10°S$ to $10°N$ and has a zonal width of $50°$. According to (**??**), dx_{ij} then takes values from $27.4\, km$ at the meridional boundaries to $27.8\, km$ at the equator.

3.1.2 Vertical

As mentioned earlier, the vertical coordinate is replaced by the vertical normal modes, such that no explicit $z-$derivatives appear in (**??**). The density field used to compute the vertical normal modes and thus the modes themselves contain all discretized vertical structure. In order to solve the eigenvalue problem for Λ, a buoyancy frequency profile obtained from observational data was used (Shown in Figure **??**). It was averaged from data obtained during five cruises, all near the equator at $23°W$ (?, ?). As stated in (**??**), N^2 is defined through the $(t, x, y)-$independent background density field, which is an assumption not guaranteed for the observations. However, seasonal and horizontal variations are not too grave in the tropical Atlantic (?, ?) and the used profile produces a sufficiently realistic background density field.

The profile in Figure **??**, its vertical grid spacing $dz = 5\, m$ and the boundary conditions in (**??**) are used to build the discrete matrix representing the operator Λ in (**??**). The corresponding eigenvalue problem (**??**) is solved using a function from the python package NumPy (?, ?). Through the high resolution of the N^2-profile with 931 levels, this also yields 931 pairs of eigenvalues λ and eigenfunctions ψ. Some properties of these vertical structure functions ψ_k are well illustrated for the second and seventh baroclinic mode in Figure **??**.
In agreement with the boundary conditions formulated in (**??**), they show zero

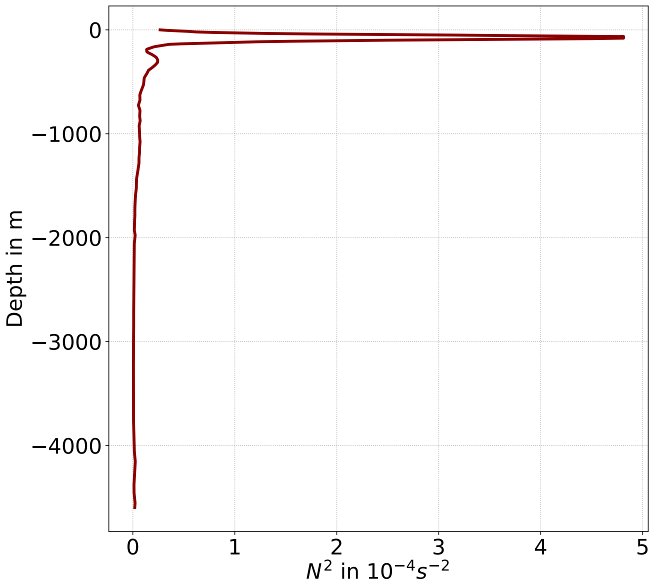

Figure 2: The N^2-profile used for the modal decomposition. The dataset has a vertical resolution of $5m$ and its maximum at a depth of $65m$.

Figure 3: The second and seventh vertical structure function derived from the eigenvalue problem in (**??**) with the N^2-profile seen in Figure **??**.

vertical derivatives at the top and bottom of the profile. Ordering the modes with increasing eigenvalues λ_k gives a decreasing equivalent depth H_k through (**??**). The distances between zero crossings get smaller, resulting in k zero crossings for the k^{th} mode. Numerical values for the equivalent depth H_k and the corresponding gravity wave speed c_k for selected modes are given in Table **??**.

mode k	1	2	3	7	16	25
H_k in cm	63.7	21.2	10.5	1.8	0.3	0.1
c_k in m/s	2.50	1.44	1.01	0.42	0.18	0.12
$\psi_k\,(z=0)$	3.54	6.29	3.47	2.91	2.63	2.45

Table 1: The values of the equivalent depth H_k, gravity wave speed c_n and the surface structure functions for selected modes.

Only the 25 most dominant eigenfunctions with the smallest eigenvalues, except for the barotropic mode with $\lambda_0 = 0$, are taken into account by the model. This resembles the resolution of the discretized Hilbert space spanned by the solutions to the Sturm-Liouville problem (**??**), which replace the vertical coordinate of the model.

3.1.3 Temporal

The discretization of the time axis is a trade-off between the size of the time step Δt and the numerical stability of the time stepping algorithm. By maximizing the time step, computational cost can be minimized. But only to the extent that the numerical solution stays numerically stable. This is the case when the error of the numerical approximation to the true solution stays bounded with increasing simulation time (?, ?).

When defining the state of the system (**??**) as $S\left(u, v, \eta\right)$ the time integration problem can be written as

$$\frac{dS}{dt} = F\left(S\right), \tag{3.10}$$

where F represents all terms defining the tendencies found in (**??**). The time axis is discretized into equidistant points in time

$$t_n = t_0 + n\,\Delta t \qquad \left(n \in \mathbb{Z}\right). \tag{3.11}$$

This allows for the numerical approximation of (**??**) through the
3rd order *Adams-Bashforth* scheme

$$S\left(t_{n+1}\right) = S\left(t_n\right) + \frac{\Delta t}{12}\left[23\,F\left(S(t_n)\right) - 16\,F\left(S(t_{n-1})\right) + 5\,F\left(S(t_{n-2})\right)\right]. \tag{3.12}$$

The scheme has an accuracy of order $\mathcal{O}(\Delta t^4)$ and requires only one function evaluation per new time step, when previous evaluations are stored (?, ?). These properties make the 3rd order *Adams-Bashforth* scheme an appropriate choice for the simulations performed for this work. Note also that options for lower order *Adams-Bashforth* schemes are implemented in the model. This is in fact necessary to compute the required three initial states from one single initial condition.

The stability of the 3rd-order Adams-Bashforth scheme is analyzed in detail by ? (?). For oscillatory solutions numerical stability of the scheme requires that

$$\|\omega\,\Delta t\| < 0.72, \tag{3.13}$$

where ω is the angular frequency in the oscillation equation

$$\frac{dS}{dt} = i\omega S. \tag{3.14}$$

The numerical representation of the shortest resolvable wavelength $2\,\Delta x$ with $\omega = c\frac{\pi}{\Delta x}$ thus demands that (**??**) is fulfilled. The fastest gravity wave resolved by the model is associated with the first baroclinic mode $c_1 \approx 2.5m/s$ and the smallest horizontal grid spacing is found at the meridional boundaries with $dx \approx 27.4km$. These values allow for the used time step of $\Delta t = 1095\,s$.

For the dissipative terms of the model such as the vertical mixing or advection terms, numerical stability requires that

$$\|\kappa\,\Delta t\| < 0.55, \tag{3.15}$$

where kappa is the non-negative damping coefficient of the friction equation

$$\frac{dS}{dt} = -\kappa S. \tag{3.16}$$

This condition is generously fulfilled by the values found in the tensors $\mathcal{P}$ and $\mathcal{Q}$ for both, uniform and McCrearys vertical mixing.